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The statistics of dimers on a nonplanar lattice

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Abstract. The dimer problem on the covering lattice of the plane square lattice is shown to be isomorphic with an eight vertex model, and can be solved exactly when conjugate dimer activities are equal. Transitions in a more general dimer problem, in which the dimers interact, are investigated. The dimer problem on the 'bathroom tile' lattice is also shown to be isomorphic with an eight vertex model, but in this case the problem can be solved only when the free-fermion condition is satisfied.

1. Introduction

It is well known that the generating function for close-packed dimer configurations on planar lattices can be calculated exactly by use of the Pfaffian enumeration technique (Temperley and Fisher 1961, Fisher 1961, Kasteleyn 1961, 1963, 1967, Montroll 1964). Unfortunately, with the exception of certain rather special cases (Kasteleyn 1963, 1967), the Pfaffian method fails when the lattice is nonplanar. As we shall demonstrate, however, exact results can be obtained for the pure (close-packed) dimer problem on a lattice with crossed bonds by utilizing an isomorphism with the eight vertex lattice problem (Baxter 1972).

The lattice concerned is the covering lattice $S^{(c)}$ of the plane square lattice S: a vertex $v_i^{(c)}$ is situated at the centre of every edge e_i of S, and two such vertices $v_i^{(c)}$ and $v_j^{(c)}$ are connected by an edge $e_{ij}^{(c)}$ in $S^{(c)}$ if, and only if, their two corresponding edges e_i and e_j in S have a common vertex in S. $S^{(c)}$ can be regarded as a plane square lattice augmented by the diagonals of alternate squares (see figure 1(a)).

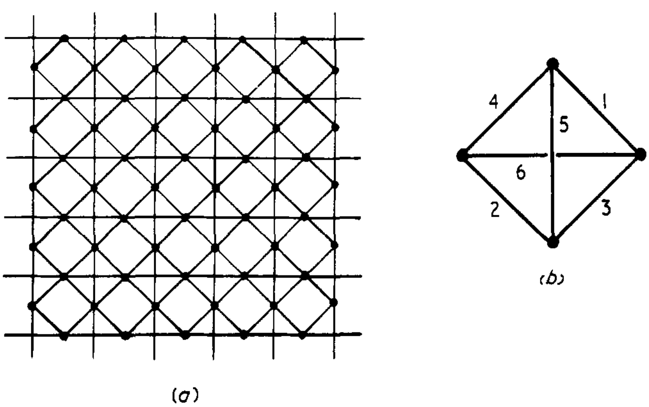

Figure 1. (a) The covering lattice $S^{(c)}$ of the plane square lattice S. (b) A 'city' on four vertices.

The eight vertex model consists of a plane square lattice with the edges directed in such a way that even numbers of arrows enter and leave every vertex. There are then eight different permitted arrow arrangements at a vertex (figure 2), and if a weight ω_k is associated with vertex type (k), the partition function of the eight vertex model on a lattice of M rows and N columns is

$$Z_{(8\text{V})} = \sum_{\text{aac}} \prod_{k=1}^{8} \omega_k^{n_k} = \sum_{\text{aac}} \prod_{i=1}^{M} \prod_{j=1}^{N} \omega_{ij}. \tag{1}$$

Here n_k is the number of vertices of type (k) in a configuration of arrows, ω_{ij} is the weight associated with the vertex at lattice site (ij) and the sum is over all allowed arrow configurations on S.

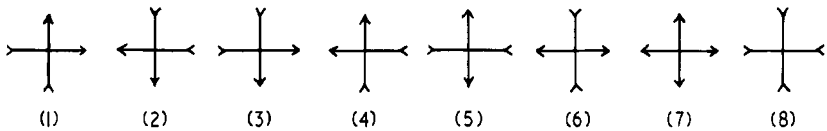

Figure 2. The eight arrow arrangements allowed at a vertex of S.

2. Equivalence between the dimer and eight vertex problems

$S^{(c)}$ can be though of as a square array of complete star graphs, or cities, on four vertices (figure 1(*b*)), such that: (*a*) every city is joined to a neighbouring city at a vertex; and (*b*) every city on $S^{(c)}$ is uniquely associated with a vertex of S. A dimer configuration on $S^{(c)}$ can therefore be described uniquely by specifying the arrangements of dimers on every city; there are ten such city dimer arrangements (figure 3). If we associate an energy ϵ_i and an activity $z_i = \exp(-\epsilon_i/kT)$ with the presence of a dimer on edge i of a city (figure 1(*b*)), then the generating function for dimer coverings of $S^{(c)}$ is

$$Z_{(\text{d})} = \sum_{\text{aac}}' \prod_{i=1}^{M} \prod_{j=1}^{N} z_{ij} \tag{2}$$

where z_{ij} is the activity of the dimer arrangement on the city associated with vertex (ij) of S and the (primed) sum runs through all close-packed dimer configurations on $S^{(c)}$.

We can now obtain a correspondence between arrow arrangements at the vertices of S and dimer coverings of the cities of $S^{(c)}$. If a dimer lies on an edge $e_{ij}^{(c)}$ between vertices

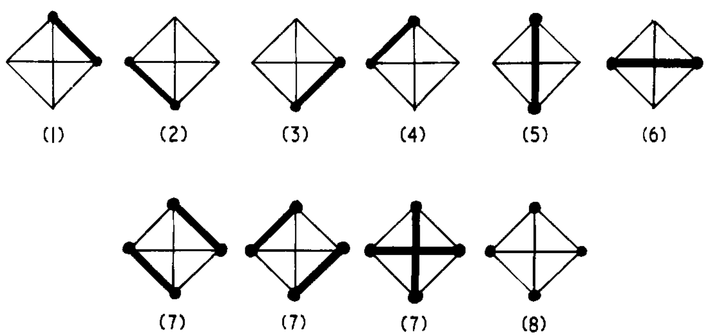

Figure 3. The dimer arrangements allowed on a city of $S^{(c)}$.

$v_i^{(c)}$ and $v_j^{(c)}$ of the city associated with a vertex v of S, place arrows pointing away from v on the edges e_i and e_j, and towards v on the remaining edges of S incident with v. It is easy to convince oneself that every dimer covering of $S^{(c)}$ can be associated uniquely with an eight vertex configuration on S. The converse is not true, however, for to every type (7) vertex (figure 2) there are three possible dimer arrangements (figure 3). The numbering of the arrangements in figure 2 and figure 3 is in accordance with the above equivalence between arrow and dimer arrangements.

From equations (1) and (2) it is evident that $Z_{(8V)} = Z_{(d)}$ provided we make the identifications

$$\begin{aligned} \omega_k &= z_k, \qquad k = 1, 2, \ldots, 6; \\ \omega_7 &= z_1 z_2 + z_3 z_4 + z_5 z_6; \qquad \omega_8 = 1. \end{aligned} \tag{3}$$

If periodic boundary conditions are imposed, there are the same numbers of type (7) and type (8) vertices, and also the same numbers of type (5) and type (6) vertices, so without loss of generality we can replace ω_5 and ω_6 by $(z_5 z_6)^{1/2} \equiv c$, say and ω_7 and ω_8 by $(z_1 z_2 + z_3 z_4 + z_5 z_6)^{1/2}$. At present the eight vertex model can be solved only when the restrictions $\omega_1 = \omega_2 \equiv a$ and $\omega_3 = \omega_4 \equiv b$ are imposed and we shall henceforth consider these cases only.

The dimer problem can be generalized slightly by allowing the dimers to interact with one another. Specifically, if two dimers simultaneously occupy edges of the same city, they acquire the interaction energy h_a, h_b or h_c according as edges 1 and 2, 3 and 4 or 5 and 6 of the city are occupied. With the definitions

$$\alpha \equiv \mathrm{e}^{-h_a/kT}, \qquad \beta \equiv \mathrm{e}^{-h_b/kT}, \qquad \gamma \equiv \mathrm{e}^{-h_c/kT}, \tag{4}$$

the interacting dimer problem on $S^{(c)}$ is isomorphic with an eight vertex model with weights

$$\begin{aligned} \omega_1 &= \omega_2 = z_1 = z_2 = a, \qquad \omega_3 = \omega_4 = z_3 = z_4 = b \\ \omega_5 &= \omega_6 = z_5 = z_6 = c, \qquad \omega_7 = \omega_8 = (a^2\alpha + b^2\beta + c^2\gamma)^{1/2}. \end{aligned} \tag{5}$$

If $\alpha = \beta = \gamma = 1$ and $c = 0$, these weights reduce to those obtained by Baxter (1972) for the dimer problem on S.

An interesting special case occurs if $\alpha = \beta = 0$, that is if two dimers may not simultaneously occupy the boundary edges of a city. The choice $\epsilon_1 = \epsilon_2 = -J - J'$, $\epsilon_3 = \epsilon_4 = J + J'$, $\epsilon_5 = \epsilon_6 = J' - J$ and $h_c = -4J' + 4J$ then yields an eight vertex problem equivalent to a nearest-neighbour Ising model on S, with interaction parameters J and J' (Wu 1969, Baxter 1972).

3. Phase transitions

Baxter has given rules for the determination of phase transitions in the eight vertex model. At a transition, the singular part of the free energy is proportional to

$$\cot(\pi^2/2\mu)|T - T_c|^{\pi/\mu} \tag{6a}$$

or, if $\pi/2\mu \equiv m$ is an integer, to

$$2/\pi(T - T_c)^{2m} \ln|T - T_c| \tag{6b}$$

where μ is a function of the vertex weights at the transition point.

Following Baxter, we define

$$w_1 = \tfrac{1}{2}\{(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}+c\} \qquad w_2 = \tfrac{1}{2}|(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}-c|$$
$$w_3 = \tfrac{1}{2}|a-b| \qquad w_4 = \tfrac{1}{2}(a+b) \tag{7}$$

and we assume that $a, b, c, \alpha, \beta, \gamma$ are all non-negative. There are then six possible regimes, namely:

$$\begin{array}{llll} (1) & w_4 \geqslant w_1 \geqslant w_2 \geqslant w_3 & (2) & w_1 \geqslant w_4 \geqslant w_3 \geqslant w_2 \\ (3) & w_1 \geqslant w_2 \geqslant w_4 \geqslant w_3 & (4) & w_1 \geqslant w_4 \geqslant w_2 \geqslant w_3 \\ (5) & w_4 \geqslant w_3 \geqslant w_1 \geqslant w_2 & (6) & w_4 \geqslant w_1 \geqslant w_3 \geqslant w_2 \end{array} \tag{8}$$

and a transition occurs in a regime when the middle two w's become equal.

The transitions are:

Regime (1).

$$\text{(i)} \quad c = 0, \qquad \cos\mu = \frac{a^2(\alpha-1)+b^2(\beta-1)}{2ab}.$$

Hence the dimer problem on S is a singular case of the dimer problem on $S^{(c)}$; if the dimers do not interact, $\alpha = \beta = 1$ and $\mu = \pi/2$, and the problem can be solved by the Pfaffian method.

$$\text{(ii)} \quad a^2\alpha+b^2\beta+c^2\gamma = 0, \qquad \cos\mu = \frac{c^2-a^2-b^2}{2ab} = -\Delta$$

where Δ is the six vertex model parameter (Lieb and Wu 1972, and references therein). We infer that the ice-like models with Δ between -1 and $+1$ can be regarded as singular cases of the eight vertex models and that the parameter Δ reflects the nature of the transition from the eight vertex to six vertex models. A similar observation has been made by Sutherland (1970). Singularities occur in the six vertex model when $\Delta = \pm 1$, that is at the extremes of the range of Δ over which the ice-like models can be regarded as eight vertex models at a transition, and these further singularities correspond to the equality of more than two of the w's (Baxter 1972).

Regime (2).

$$\text{(i)} \quad a = 0, \qquad \cos\mu = \frac{b^2(1-\beta)-c^2(1+\gamma)}{2c(b^2\beta+c^2\gamma)^{1/2}}$$

$$\text{(ii)} \quad b = 0, \qquad \cos\mu = \frac{a^2(1-\alpha)-c^2(1+\gamma)}{2c(a^2\alpha+c^2\gamma)^{1/2}}.$$

In contrast to the transition at $c = 0$, μ depends upon b and c (or a and c) even when the dimers do not interact.

Regimes (3) and (4).

$$|(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}-c| = a+b,$$

$$\cos\mu = \frac{ab-c(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}}{ab+c(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}}.$$

Regimes (5) and (6).

$$(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}+c = |a-b|,$$

$$\cos\mu = \frac{c(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}-ab}{c(a^2\alpha+b^2\beta+c^2\gamma)^{1/2}+ab}.$$

These transitions in regimes 3, 4, 5 and 6 include the Ising model transitions, for which $\mu = \pi/2$.

In addition, if more than two w's become equal in any regime, we obtain transitions analogous to the $\Delta = \pm 1$ transitions in the ice-like models.

4. Dimer coverings of the terminal lattice $S^{(T)}$

The terminal lattice (or 'bathroom tile' lattice) $S^{(T)}$ consists of a square lattice in which every vertex is replaced by a city (figure 4). The possible arrangements of close-packed dimers on a city and its adjacent intercity edges are shown in figure 5. We obtain an

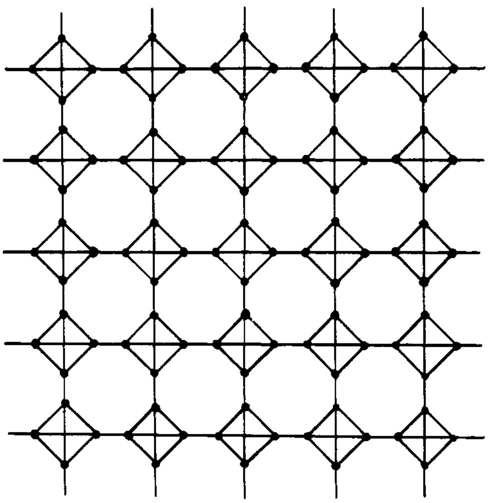

Figure 4. The terminal, or 'bathroom tile' lattice $S^{(T)}$.

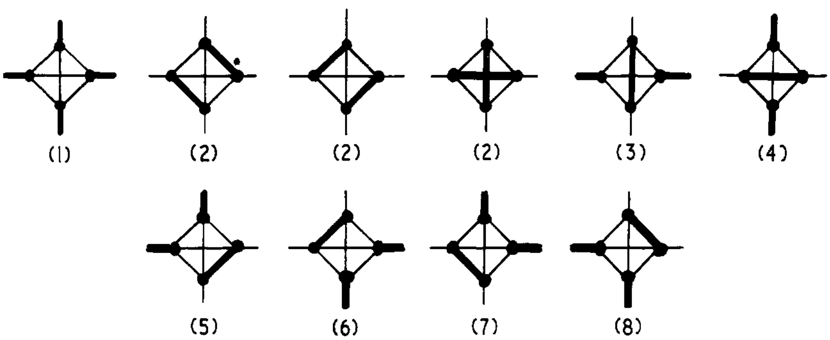

Figure 5. The dimer arrangements allowed at a city of $S^{(T)}$.

equivalence between dimer configurations on $S^{(T)}$ and arrow configurations on S as follows: if a dimer is situated on a horizontal (vertical) intercity edge of $S^{(T)}$ place an arrow pointing to the right (upward) on the corresponding edge of S, otherwise place an arrow pointing to the left (downward) on that edge. Then every dimer configuration on $S^{(T)}$ corresponds uniquely with an eight vertex configuration on S; the converse is not true, however, for to every type (2) vertex (figure 2) there are three possible dimer arrangements.

Dimer activities x and y are ascribed to horizontal and vertical intercity edges and activities a, b, c and interactions α, β, γ are introduced for city dimers, as in the covering lattice case. Then the dimer problem on $S^{(T)}$ is equivalent to an eight vertex model with weights

$$\omega_1 = xy, \qquad \omega_2 = a^2\alpha + b^2\beta + c^2\gamma, \qquad \omega_3 = cx, \qquad \omega_4 = cy,$$
$$\omega_5 = \omega_6 = b(xy)^{1/2}, \qquad \omega_7 = \omega_8 = a(xy)^{1/2}. \tag{9}$$

Since $\omega_1 \neq \omega_2$ this eight vertex model cannot in general be solved; however it is soluble if

$$\omega_1\omega_2 + \omega_3\omega_4 = \omega_5\omega_6 + \omega_7\omega_8, \tag{10}$$

which is the free-fermion condition (Fan and Wu 1969, 1970, Lieb and Wu 1972). The condition (10) applied to the weights in (9) yields

$$\omega_2 = a^2 + b^2 - c^2; \tag{11}$$

if, in addition to (11) we choose $a = b = c = 1$, the free-fermion model is isomorphic with the nearest-neighbour Ising model on S (Kasteleyn 1963, 1967).

The parameters in (9) can, in fact, be chosen so that the dimer problem is equivalent to a square lattice Ising model with first- and second-neighbour interactions together with four spin forces (Wu 1971, Lieb and Wu 1972). A similar procedure is possible for the dimer problem on the covering lattice if conjugate dimer activities are allowed to be unequal.

5. Conclusion

The problem of interacting dimers on the covering lattice of the plane square lattice is isomorphic with an eight vertex model, and can be solved exactly provided conjugate dimer activities are equal. The dimer problem on the 'bathroom tile' lattice is soluble only if the vertex weights of the corresponding eight vertex model satisfy the free-fermion condition. Both of these dimer problems reduce to the square lattice Ising model with many-body interactions under appropriate choices of parameters. Furthermore the ice models and the plane square lattice dimer problems can both be regarded as the transition regions of more general models.

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